Optimal Packing Circular Cylinders into a Cylindrical Container Taking into Account Behavior Constraints

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1. Problem formulation
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Let \( \{C_i, i = 1, 2, \ldots, N\} \) be a set of circular cylinders. Each cylinder \( C_i \) given by its radius \( r_i \), height \( h_i \) and mass \( m_i \). Let \( \Omega \) be a container of cylindrical form of radius \( R \) and height \( 2H \).

We denote container \( \Omega \) with the set of cylinders inside the container by \( \Omega^A \).
Problem

Pack the set of cylinders \( \{C_i, i = 1, 2, \ldots, N\} \) into a container \( \Omega \) of minimal radius taking into account mechanical behavior constraints (balance, inertia moments, stability) of system \( \Omega^A \).
These problems have a wide spectrum of applications in space engineering for satellite modeling [Fasano, Pinter (2012)].

The packing problem is considered in the paper

Chao Che, Yi-shou Wang, Hong-fei Teng. Test problems for quasi-satellite packing: Cylinders packing with behavior constraints and all the optimal solutions known. (Online), (2008)

In order to solve the problem authors use a heuristic algorithm.
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We present a mathematical model of the problem as a constraint optimization problem using phi-function technique

\[ F(u^*) = \min_{u \in W \subset \mathbb{R}^{3N+1}} F(u), \] (1)

\[ W = \left\{ u \in \mathbb{R}^{3N+1} : \Phi_k(u) \geq 0, k = 1, \ldots, N(N+1)/2, \right. \]
\[ \left. G_1(u) \geq 0, G_2(u) \geq 0, G_3(u) \geq 0, \right. \]
\[ R \geq r_i, i = 1, \ldots, N, \] (2)

where
\[ F(u) = R \] is an objective function,
\[ u = (u_1, u_2, \ldots, u_N, R) \] is a vector of variables,
\[ u_i = (x_i, y_i, z_i) \] is a vector of placement parameters of cylinder,
\[ W \] is a feasible region
The feasible region $W$ is formed by **placement** constraints and **behavior** constraints.

**Placement** constraints use phi-functions for description of non-overlapping and containment constraints

$$
\Phi_{ij}^{CC} \geq 0, \; i > j = 1, \ldots, N - 1, \quad \Phi_i^{\Omega^*C} \geq 0, \; i = 1, \ldots, N
$$

where $\Phi_{ij}^{CC}$ is phi-function for two cylinders $C_i$ and $C_j$, $\Phi_i^{\Omega^*C}$ is phi-function for cylinder $C_i$ and object $\Omega^* = E^3 \setminus \text{int} \Omega$. 
Behavior constraints include:

- balance constraints $G_1(u) \geq 0$,
- inertia moment constraints $G_2(u) \geq 0$,
- stability constraints $G_3(u) \geq 0$.

Let us consider behavior constraints in details.
Center of mass \((x_c, y_c, z_c)\) of the set \(\{C_i, i \in I_N\}\) is defined as:

\[
x_c = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}, \quad y_c = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i}, \quad z_c = \frac{\sum_{i=1}^{N} m_i z_i}{\sum_{i=1}^{N} m_i}.
\]

\((x_e, y_e, z_e)\) is a center of mass of \(\Omega^A\), which coincide with the symmetry centre of container \(\Omega\). We assume that \((x_e, y_e, z_e) = (0, 0, 0)\).
Balance constraints have the following form:

\[ G_1(u) \geq 0, \]

where

\[ G_1(u) = \min\{g_1(u), g_2(u), g_3(u)\}, \]
\[ g_1(u) = \min\{- (x_e - x_c) + \Delta x_c, (x_e - x_c) + \Delta x_c\}, \]
\[ g_2(u) = \min\{- (y_e - y_c) + \Delta y_c, (y_e - y_c) + \Delta y_c\}, \]
\[ g_3(u) = \min\{- (z_e - z_c) + \Delta z_c, (z_e - z_c) + \Delta z_c\}, \]

\((\Delta x_c, \Delta y_c, \Delta z_c) - \) allowable deviations from point \((x_e, y_e, z_e)\) of center of mass of \(\Omega^A\).
Mathematical model

Inertia moment constraints

Inertia moments $J_x(u), J_y(u), J_z(u)$ of system $\Omega^A$ are defined as

$$J_x(u) = \sum_{i=1}^{N} J''_{xi} + \sum_{i=1}^{N} m_i(y_i^2 + z_i^2) - (y_c^2 + z_c^2) \sum_{i=1}^{N} m_i,$$

$$J_y(u) = \sum_{i=1}^{N} J''_{yi} + \sum_{i=1}^{N} m_i(x_i^2 + z_i^2) - (x_c^2 + z_c^2) \sum_{i=1}^{N} m_i,$$

$$J_z(u) = \sum_{i=1}^{N} J''_{zi} + \sum_{i=1}^{N} m_i(x_i^2 + y_i^2) - (x_c^2 + y_c^2) \sum_{i=1}^{N} m_i.$$

where

$$J''_{xi} = J''_{yi} = \frac{1}{12} m_i(3r_i^2 + 4h_i^2), J''_{zi} = \frac{1}{2} m_i r_i^2.$$
Inertia moment constraints have the following form:

\[ G_2(u) \geq 0, \]

where

\[ G_2(u) = \min\{g_4(u), g_5(u), g_6(u)\}, \]

\[ g_4(u) = \min\{-J_x(u) + \Delta J_x, J_x(u) + \Delta J_x\}, \]

\[ g_5(u) = \min\{-J_y(u) + \Delta J_y, J_y(u) + \Delta J_y\}, \]

\[ g_6(u) = \min\{-J_z(u) + \Delta J_z, J_z(u) + \Delta J_z\}, \]

\((\Delta J_x, \Delta J_y, \Delta J_z)\) – allowable deviations from inertia moment of \(\Omega^A\)
Angle deviations $\varphi_x(u), \varphi_y(u), \varphi_z(u)$ of the main inertia axis of the system from axis of the fixed coordinate system are defined by the following relations:

$$\varphi_x(u) = \frac{1}{2} \arctg \left( \frac{2J_{xy}(u)}{J_y(u) - J_x(u)} \right), \quad J_{xy}(u) = \sum_{i=1}^{N} m_i x_i y_i - x_c y_c \sum_{i=1}^{N} m_i,$$

$$\varphi_y(u) = \frac{1}{2} \arctg \left( \frac{2J_{yz}(u)}{J_z(u) - J_y(u)} \right), \quad J_{yz}(u) = \sum_{i=1}^{N} m_i y_i z_i - y_c z_c \sum_{i=1}^{N} m_i,$$

$$\varphi_z(u) = \frac{1}{2} \arctg \left( \frac{2J_{xz}(u)}{J_z(u) - J_x(u)} \right), \quad J_{xz}(u) = \sum_{i=1}^{N} m_i x_i y_i - x_c y_c \sum_{i=1}^{N} m_i.$$
Stability constraints have the following form:

\[ G_3(u) \geq 0, \]

where

\[ G_3(u) = \min\{g_7(u), g_8(u), g_9(u)\}, \]

\[ g_7(u) = \min\{-\varphi_x(u) + \Delta \varphi_x, \varphi_x(u) + \Delta \varphi_x\}, \]

\[ g_8(u) = \min(-\varphi_y(u) + \Delta \varphi_y, \varphi_y(u) + \Delta \varphi_y), \]

\[ g_9(u) = \min(-\varphi_z(u) + \Delta \varphi_z, \varphi_z(u) + \Delta \varphi_z), \]

\(\Delta \varphi_x, \Delta \varphi_y, \Delta \varphi_z\) are allowable errors of angles

\(\varphi_x(u), \varphi_y(u), \varphi_z(u)\)
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Solution method

**Step 1.** We generate a new function

\[
f(u) = R + P_1 \sum_{k=1}^{n} \max\{0, -\Phi_k\} + P_2 \sum_{k=n+1}^{n+18} \max\{0, -g_k\} + P_3 \max\{0, -R + \max_{i=1,...,N} r_i\},
\]

by means of non-smooth penalty \( P_1, P_2, P_3, n = N(N+1)/2 \), \( \Phi_k \) are given phi-functions and \( g_k \) are given behavior functions

**Step 2.** We reduce problem (1)–(2) to the following non-constrained nonsmooth optimization problem

\[
\min_{u \in \mathbb{E}^{3N+1}} f(u).
\]
In order to realize the model (3) we

1) generate a set of random starting points

2) apply Shor’s r-algorithm\(^1\) for each starting point to search for a local minima

Computational results

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Computational results

Simple test problem

We show our results with help of the simple test problem:

- $N = 5$
- $H = 1$, $h_i = 1$, $i = 1, \ldots, 5$

Cylinders have different radii and masses:

- $r_1 = 0.1$, $r_2 = 0.2$, $r_3 = 0.3$, $r_4 = 0.5$, $r_5 = 0.8$
- $m_1 = 0.0785$, $m_2 = 0.314$, $m_3 = 0.7065$, $m_4 = 1.9625$, $m_5 = 5.024$

System behavior parameters:

- $(x_e, y_e, z_e) = (0, 0, 0)$
- $(\Delta x_c, \Delta y_c, \Delta z_c) = (0.0001, 0.0001, 0.0001)$
- $(\Delta J_x, \Delta J_y, \Delta J_z) = (5, 5, 5)$
Computational results

Optimal placement of cylinders (view from above)

\[ F(u^*) = R^* = 1.3000 \] is the global minimum of the problem without behavior constraints

\[ F(u_1^*) = R^* = 1.3161 \] is the global minimum of the problem with balance constraints
Computational results

Optimal placement of cylinders (view from above)

\[ F(u_2^*) = R^* = 1.3161 \] is the global minimum of the problem with balance constraints.

\[ F(u^*) = R^* = 1.3625 \] is the global minimum of the problem with balance and inertia moment constraints.
to test our algorithm for medium and large size problems
Thanks

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Question?

Thank you for your attention
phi-functions for 3D-case

\[
\Phi_{ij}^{CC} = \max\{(x_j - x_i)^2 + (y_j - y_i)^2 - (r_i + r_j)^2, \\
z - (h_i + h_j), -z - (h_i + h_j)\},
\]

\( (1) \)

\[
\Phi_{i}^{CC} = \min\{-x_i^2 - y_i^2 + (R - r_i)^2, -z + (H - h_i), \\
z + (H - h_i)\}
\]

\( (2) \)

phi-functions for 2D-case \((H = h_i)\)

\[
\Phi_{ij}^{CC} = (x_j - x_i)^2 + (y_j - y_i)^2 - (r_i + r_j)^2,
\]

\( (3) \)

\[
\Phi_{i}^{CC} = -x_i^2 - y_i^2 + (R - r_i)^2,
\]

\( (4) \)